Common issues from HW:

1.3.3. Make sure you didn't assume inf A exists when you argue (a). Should prove:

(1) $\forall a \in A$, $\sup B \le a$

(2) $\forall b > \sup B$, $\exists a \in A$, $a < b$.

Therefore $\inf A$ exists and is equal to $\sup B$.

If you assumed $\inf A$ exists in (a), then addition argument will be needed for (b) to avoid circular reasoning.

Summary of Chap 2.

(1) Definition of sequential limit. Uniqueness.

(2) Algebraic Limit Theorem; Order limit theorem.

(3) Monotone Convergence Theorem:

$$(a_n) \text{ monotone} \wedge (a_n) \text{ bounded} \Rightarrow \lim_{n\to\infty} a_n \text{ exists}$$

Pf: Existence of sup. (aka. Axiom of Completeness)

(4) Series. Partial Sum. Convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}. \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

Cauchy Condensation Test:

$(b_n) \searrow$, $b_n \ge 0$, $\forall n \in \mathbb{Z}_+$

$\sum_{n=1}^{\infty} b_n$ converges iff $\sum 2^n b_{2^n}$ converges

p-test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$.

(5) Subsequence: (a_{n_k})

Thm: $(a_n) \to a \Leftrightarrow \forall$ subseq. $(a_{n_k}) \to a$

Variant: If $(a_{m_k}), (a_{n_k})$ are two subsequences, $\lim_{k\to\infty} a_{m_k} \neq \lim_{k\to\infty} a_{n_k}$

then (a_n) diverges.

Limit superior $\limsup_{n\to\infty} a_n$. Limit inferior $\liminf_{n\to\infty}$.

Lemma: $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$

Thm: $\lim a_n = a \Leftrightarrow \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$.

(6) Bolzano-Weierstrass Theorem:

Every bounded sequence contains a convergent subsequence.

Pf: Nested Interval Property.

(7) Cauchy Sequence: $\forall \varepsilon > 0, \exists N \in \mathbb{Z}_+, \forall m, n > N, |a_m - a_n| < \varepsilon$

Cauchy Criterion: (a_n) converges $\Leftrightarrow$ (a_n) is Cauchy.

Pf: $\Rightarrow$ triangle ineq. $\Leftarrow$ Bolzano-Weierstrass.

(8) Completeness:

If Archimedean Property is assumed, then

$$A_o C \Leftrightarrow NIP \Leftrightarrow MCT \Leftrightarrow BW \Leftrightarrow CC.$$

Read: Real Analysis in Reverse.

(9) Series Version of

- Algebraic Limit Theorem: $\sum_{k=1}^{\infty} a_k = A$, $\sum_{k=1}^{\infty} b_k = B$

(i) $\sum_{k=1}^{\infty} ca_k = cA$, $\forall c \in \mathbb{R}$. (ii) $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

- Cauchy Criterion:

$\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow \forall \varepsilon>0, \exists N\in\mathbb{Z}_+, \forall n>m>N, \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$

Pf: Express in partial sums.

- Order Limit Theorem: Comparison test.

If $(a_k), (b_k)$ satisfies $0 \le a_k \le b_k$

(i) $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges

(ii) $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} b_k$ diverges

Pf: Cauchy criterion.

(10) Absolute Convergence Test: $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum a_n$ converges.

Thm: $\sum_{n=1}^{\infty} a_n$ converges absolutely $\Rightarrow$ Any rearrangement of the series converges to the same limit

Alternating Series Test: $(a_n) \searrow 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$

Conditional Convergence: $\sum_{n=1}^{\infty} a_n$ exists but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Riemann Series Theorem: If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\forall M\in[-\infty, +\infty], \exists \sigma: \mathbb{Z}_+ \leftrightarrow \mathbb{Z}_+, \sum_{n=1}^{\infty} a_{\sigma(n)} = M$.

HW graded: 2.2.2a, 2.3.2b*, 2.3.3, 2.3.7*, 2.3.10

My solution to 2.3.2b : $(x_n) \to 2 \Rightarrow \left(\frac{1}{x_n}\right) \to \frac{1}{2}$

$(x_n) \to 2 \Rightarrow \forall \varepsilon > 0,\ \exists N_1 > 0,\ \forall n > N_1,\ |x_n - 2| < \varepsilon$

Pick $\varepsilon = \frac{3}{2}$, then $\exists N_2 > 0,\ \forall n > N_2,\ x_n - 2 > \frac{3}{2} \Rightarrow 2x_n > 1$
$\Rightarrow \left|\frac{1}{2x_n}\right| < 1$

Therefore for $n > \max(N_1, N_2)$,

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = \frac{1}{|2x_n|}\,|x_n - 2| < 1 \cdot \varepsilon = \varepsilon.$$

So $\lim_{n\to\infty} \frac{1}{x_n} = 2$.

My solution to 2.3.7.

(a) $x_n = n$, $y_n = -n$, $x_n + y_n = 0$. $(x_n + y_n) \to 0$ but $(x_n), (y_n)$ diverge.

(b) If (a_n) and $(a_n + b_n)$ converges, then by the algebraic limit theorem $(b_n) = (a_n + b_n) - (a_n)$ converges. So it's impossible to find an example that satisfies the statement.

(c). $b_n = n$. Then $\left(\frac{1}{b_n}\right)$ converges with $b_n \neq 0, \forall n$, but (b_n) diverges

(d) If (b_n) converges then (b_n) is bounded.
If $(a_n - b_n)$ is bounded then since $|a_n| \leq |a_n - b_n| + |b_n|$, (a_n) will be bounded. So it's impossible to find an example that satisfies the statement.

(e) $a_n = \frac{1}{n^2}$, $b_n = n$, then both (a_n) and $(a_n b_n)$ converges. but b_n diverges

(c),(e) explains : $(a_n) \to a,\ (b_n) \to b \boxed{\neq 0} \Rightarrow \left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$